

Gravitational dynamics in $s + 1 + 1$ dimensions II. Hamiltonian theory

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We develop a Hamiltonian formalism of brane-world gravity, which singles out two preferred, mutually orthogonal directions. One is a unit twist-free field of spatial vectors with integral lines intersecting perpendicularly the brane. The other is a temporal vector field with respect to which we perform the Arnowitt-Deser-Misner decomposition of the Einstein-Hilbert Lagrangian. The gravitational variables arise from the projections of the spatial metric and their canonically conjugated momenta as tensorial, vectorial and scalar quantities defined on the family of hypersurfaces containing the brane. They represent the gravitons, a gravi-photon and a gravi-scalar, respectively. From the action we derive the canonical evolution equations and the constraints for these gravitational degrees of freedom both on the brane and outside it. By integrating across the brane, the dynamics also generates the tensorial and scalar projection of the Lanczos equation. The vectorial projection of the Lanczos equation arises in a similar way from the diffeomorphism constraint. Both the graviton and the gravi-scalar are continuous across the brane, however the momentum of the gravi-vector has a jump, related to the energy transport (heat flow) on the brane.

I. INTRODUCTION

The Hamiltonian theory of general relativity is based on the Arnowitt-Deser-Misner (ADM) decomposition [1], i.e. slicings of the four dimensional (4d) space-time manifold with a family of spacelike three-dimensional (3d) hypersurfaces.

The slicing method with respect to a timelike hypersurface (the brane) has been also proven efficient in brane world models which interpret our traditional 4d space-time as a brane embedded in a 5-dimensional (5d) manifold, the bulk (for a review see [2]). In the 4+1 decomposition of the bulk, various geometrical projections of the 5d gravity on the brane were used together with the Lanczos-Sen-Darmois-Israel (LSDI) matching conditions [3]-[6] in order to derive the effective Einstein equation on the brane [7]-[8]. In these approaches the 4d space-time manifold was treated covariantly. In the covariant approach however it is not obvious how to describe time evolution, and introduce the very concept of the (many-fingered) time function, which is the first step in developing a Hamiltonian formulation of the brane world scenario.

In a previous paper [9] (hereafter quoted as Paper I), we have discussed a two-fold slicing of the bulk, both with respect to a timelike and a spacelike foliation. One of them was meant to contain the brane, the other the constant time slices Σ_t (referring to $t=\text{const}$) of the bulk, necessary for Hamiltonian evolution. The result was a $(3 + 1 + 1)$ -decomposition of the bulk, which resulted in a break-up of the brane into space and time. The formalism was given in fact for the more generic case of $s + 1 + 1$ dimensions, therefore it is applicable for general relativistic situations as well, whenever any space-like direction should be singled out for one reason or another.

In this framework the gravitational degrees of freedom are represented by new variables. These are the spatial metric g_{ab} on the brane, the shift vector M^a and lapse

function M associated with the brane normal; together with the tensorial, vectorial and scalar projections of the extrinsic curvature associated with the temporal normal. These can be thought of as gravi-tensorial, gravi-vectorial and gravi-scalar degrees of freedom, the extrinsic curvatures representing their generalized velocities.

In Paper I we have given the decomposition of all geometric quantities with respect to the two normals, together with the evolution equations of the gravi-tensorial, gravi-vectorial and gravi-scalar degrees of freedom in the velocity phase-space. A set of (Lagrangian-type) constraint equations on these variables were also given. The latter should be imposed on any set of initial data on a spacelike section of the brane.

However Paper I did not contain any reference to canonical momenta and the phase-space of gravitational degrees of freedom. Neither did it address the variational principle leading to the Hamiltonian evolution equations, together with the super-Hamiltonian and super-momentum (diffeomorphism) constraints of brane-world gravity.

It is the purpose of this paper to discuss these topics, together with the regularization issues raised by the presence of the distributional matter sources on the brane. In Sec. II we give a brief, somewhat technical account of those results presented in Paper I which are needed for our present considerations.

Sec. III contains the variational principle for vacuum gravity, with concomitant ADM and brane-bulk decompositions. We define here the momenta associated with the gravi-tensorial, gravi-vectorial and gravi-scalar degrees of freedom and perform the Legendre-transformation. Extremizing the action with respect to the gravitational phase-space variables give the Hamiltonian evolution equations in terms of variables defined on the $t = \text{const}$ space sections of the brane. Extremizing with respect to the lapse N^a and shift N (both related to the temporal evolution) gives the super-Hamiltonian

and super-momentum constraints of brane-world gravity. Both the shift N^a and super-momentum \mathcal{H}_a^G have only s components, as the off-brane component of the shift is suppressed in order to obey Frobenius' theorem. i.e. to let the brane exist as a hypersurface.¹

Matter sources both in the bulk and on the brane are discussed in Sec. IV. We show here the following commutativity property: the $(s+1+1)$ -decomposition of the energy-momentum tensor obtained by a covariant variation leads to the same result as the non-covariant variation with respect to the gravitational variables arising from the $(s+1+1)$ -decomposition. We also give here the Hamiltonian evolution in the presence of matter. Similarly as the evolution equations in the velocity phase-space given in Paper I, these Hamiltonian evolution equations contain distributional type sources.

We remedy this situation in Sec. V by a regularization procedure across the brane. We consider an interval $(-\chi_0, \chi_0)$ across the brane (lying at $\chi=0$) and integrate the super-Hamiltonian and super-momentum constraints of the total system (gravity + matter). We also integrate the Hamiltonian evolution equation of the gravi-tensorial momentum. Finally we take the thin brane limit $\chi_0 \rightarrow 0$. This method gives the scalar, vectorial and tensorial projections (with respect to Σ_t) of the Lanczos equation.

Sec. VI contains the concluding remarks. We include three Appendices, the first containing the $(s+1+1)$ -decompositions of various quantities needed throughout the paper. In the second Appendix we sketch the derivation of the projections of the Lanczos equation in the velocity phase-space. The third Appendix proves another commutativity property. While in the main paper we obtain the results by an $s+1+1$ ADM decomposition prior to applying the variational principle; the same results can be regained by first performing an $(s+1)+1$ ADM reduction, then extremizing the action and finally by a further $(s+1)$ -split.

Notation.— All quantities defined on the full $(s+2)$ -dimensional space-time and on the $(s+1)$ -dimensional brane carry a distinguishing tilde and the prefix $(s+1)$, respectively. A hat distinguishes quantities defined on the spatial $(s+1)$ -leaves. Quantities defined on the spatial sections of the brane carry no distinctive mark. For example, the metric 2-forms are denoted \tilde{g} , \hat{g} and g , respectively, while the corresponding metric-compatible connections are $\tilde{\nabla}$, \hat{D} and D . Latin indices represent abstract indices running from 0 to $(s+1)$. Vector fields in Lie-derivatives are represented by boldface characters. For example $\mathcal{L}_{\mathbf{V}}T$ denotes the Lie derivative along the in-

tegral lines of the vector field V^a . A dot denotes a derivative with respect to t , such that $\dot{T} = (\partial/\partial t)T = \mathcal{L}_{\partial/\partial t}T$.

II. THE $(s+1+1)$ -DECOMPOSITION OF SPACE-TIME

In order to develop the Hamiltonian theory of brane-world gravity from the viewpoint of an observer on the brane, a double decomposition of the bulk is necessary. We have to separate both the normal to the brane l^a and the temporal normal n^a , with respect to which the ADM decomposition will be carried out. These vector fields obey $n^a n_a = -1$, $l^a l_a = 1$ and $n^a l_a = 0$. We introduce a coordinate χ transverse to the brane and we foliate the $(s+1+1)$ -dimensional space-time by $(s+1)$ -dimensional leaves of constant χ , the brane being at $\chi=0$. The *off-brane evolution* is defined along the integral lines of

$$\left(\frac{\partial}{\partial \chi}\right)^a = M^a + M l^a, \quad (1)$$

where M^a and M represent the shift vector and lapse function associated with the off-brane evolution. They characterize the off-brane sector of gravity.

The temporal evolution is along the integral lines of

$$\left(\frac{\partial}{\partial t}\right)^a = N n^a + N^a, \quad (2)$$

where N^a and N are the familiar shift vector and lapse function related to the foliation of the $(s+1+1)$ -dimensional space-time by the $(s+1)$ -dimensional spatial leaves $t=\text{const}$. The coordinate t represents time. Note that neither $(\partial/\partial \chi)^a$ has a component along n^a , nor $(\partial/\partial t)^a$ along l^a . The first condition guarantees that $(\partial/\partial \chi)^a$ generates a purely spatial displacement. The second one is the simplest gauge-choice, which ensures that the Fröbenius theorem holds, thus l^a is hypersurface-forming (see Appendix C of Paper I).

The metric \tilde{g}_{ab} of the $(s+1+1)$ -dimensional space-time, decomposed with respect to l^a and n^a is

$$\tilde{g}_{ab} = g_{ab} - n_a n_b + l_a l_b, \quad (3)$$

where g_{ab} is the induced metric on the s -dimensional leaves $\Sigma_{t\chi}$, which represent the intersection of the $\chi=\text{const}$ and $t=\text{const}$ leaves. The components of $(s+2)$ -dimensional metric in the coordinate system adapted to the coordinates (t, χ, x^a) can be written as

$$ds^2 = -(N_a N^a - N^2)dt^2 + 2N_a dt dx^a + N_a M^a dt d\chi + g_{ab} dx^a dx^b + 2M_a dx^a d\chi + (M_a M^a + M^2)d\chi^2. \quad (4)$$

Then, as discussed in Paper I, the gravitational sector is described by $\{g_{ab}, M^a, M, N^a, N\}$ obeying $g_{ab} n^a = g_{ab} l^a = M^a n_a = M^a l_a = N^a n_a = N^a l_a = 0$. The above set has one variable less than the number of variables contained in \tilde{g}_{ab} . This is because time-evolution is

¹ While in the $s+1$ ADM decomposition the congruence $\partial/\partial t$ generating time-evolution should be twist-free in order Σ_t to exist, there is no such condition, that an $(s-1)$ -parameter subset of the congruence should form a temporal hypersurface. It is exactly this condition, which should be obeyed in brane-worlds, leading in the simplest case to the suppression of the $(s+1)^{\text{th}}$ component of the shift.

restricted to proceed along $\chi=\text{const}$ hypersurfaces. (According to Eq. (2) there is no off-brane component of the shift.)

In Paper **I** we have introduced two types of extrinsic curvatures, related to the normal vector fields n^a and l^a . Each of them could be further decomposed with respect to the other normal into tensorial, vectorial and scalar projections. These give two kinds of second fundamental forms of the leaves Σ_χ

$$K_{ab} = g^c{}_a g^d{}_b \tilde{\nabla}_c n_d, \quad L_{ab} = g^c{}_a g^d{}_b \tilde{\nabla}_c l_d, \quad (5)$$

two normal fundamental forms

$$\mathcal{K}_a = g^b{}_a l^c \tilde{\nabla}_c n_b, \quad \mathcal{L}_a = -g^b{}_a n^c \tilde{\nabla}_c l_b \quad (6)$$

and two normal fundamental scalars

$$\mathcal{K} = l^a l^b \tilde{\nabla}_a n_b, \quad \mathcal{L} = n^a n^b \tilde{\nabla}_a l_b. \quad (7)$$

From the symmetry property of the extrinsic curvatures and $n^a l_a = 0$ the condition: $\mathcal{K}_a = \mathcal{L}_a$ follows.

It was shown in Paper **I** that L_{ab} and \mathcal{L} can be expressed in terms of χ -derivatives and the covariant derivatives D_a associated with g_{ab} as

$$L_{ab} = \frac{1}{2M} \left(\frac{\partial g_{ab}}{\partial \chi} - 2D_{(a} M_{b)} \right), \quad (8a)$$

$$\mathcal{L} = -\frac{1}{MN} \left(\frac{\partial N}{\partial \chi} - M^a D_a N \right). \quad (8b)$$

Thus, L_{ab} and \mathcal{L} are related to the spatial derivatives of the gravitational variables. By contrast, the quantities K_{ab} , \mathcal{K}_a and \mathcal{K} give the time evolution of g_{ab} , M^a and M , respectively:

$$K_{ab} = \frac{1}{2N} \left(\frac{\partial g_{ab}}{\partial t} - 2D_{(a} N_{b)} \right), \quad (9a)$$

$$\mathcal{K}^a = \frac{1}{2MN} \left(\frac{\partial M^a}{\partial t} - \frac{\partial N^a}{\partial \chi} + M^b D_b N^a - N^b D_b M^a \right) \quad (9b)$$

$$\mathcal{K} = \frac{1}{MN} \left(\frac{\partial M}{\partial t} - N^a D_a M \right). \quad (9c)$$

therefore they are velocity-type variables.

Thus gravitational dynamics in the velocity phase-space can be given in terms of $\{g_{ab}, M^a, M; K_{ab}, \mathcal{K}^a, \mathcal{K}\}$. Time-evolution of $\{g_{ab}, M^a, M\}$ is expressed by Eqs. (9), while equations representing the time-evolution of $\{K_{ab}, \mathcal{K}^a, \mathcal{K}\}$ were given by Eqs. (67) of Paper **I**.

In the next section we present the Hamiltonian formulation of brane-world gravity, by passing to the momentum phase-space.

III. HAMILTONIAN EVOLUTION OF VACUUM GRAVITY IN THE BULK

Vacuum geometrodynamics (without cosmological constant) arises from the Einstein-Hilbert action

$$S^G[\tilde{g}_{ab}] = \int d^{s+2}x \mathcal{L}^G = \int d^{s+2}x \sqrt{-\tilde{g}} \tilde{R}. \quad (10)$$

The $(s+1+1)$ -decomposition of the $(s+2)$ -dimensional scalar curvature and of the metric determinant were derived in Paper **I** as Eqs. (61) and (B2), respectively. The vacuum gravitational Lagrangian density becomes

$$\begin{aligned} \mathcal{L}^G[g_{ab}, M^a, M, K_{ab}, \mathcal{K}_a, \mathcal{K}; N^a, N] = \\ NM\sqrt{g}(R - L_{ab}L^{ab} + L^2 - 2\mathcal{L}L) + 2\sqrt{g}D_a N D^a M \\ + NM\sqrt{g}(K_{ab}K^{ab} - K^2 + 2\mathcal{K}_a \mathcal{K}^a - 2\mathcal{K}\mathcal{K}) \\ - 2\tilde{\nabla}_a [NM\sqrt{g}(\alpha^a - \lambda^a - Kn^a + Ll^a)]. \end{aligned} \quad (11)$$

Here $\alpha^a = n^b \tilde{\nabla}_b n^a = N^{-1}D^a N - \mathcal{L}l^a$ and $\lambda^a = l^b \tilde{\nabla}_b l^a = -M^{-1}D^a M + \mathcal{K}n^b$ are acceleration-type quantities (the curvatures of the normal congruences n^a and l^a). By transforming (the double of) the terms quadratic in the extrinsic curvatures in the Lagrangian density (11) into time-derivative terms cf. Eqs. (9) we obtain the following advantageous expression:

$$\begin{aligned} \mathcal{L}^G[g_{ab}, M^a, M, K_{ab}, \mathcal{K}_a, \mathcal{K}; N^a, N] = \\ \sqrt{g}M [K^{ab} - (K + \mathcal{K})g^{ab}] \dot{g}_{ab} + 2\sqrt{g}\mathcal{K}_a \dot{M}^a \\ - 2\sqrt{g}K \dot{M} - N\mathcal{H}_\perp^G - N^a \mathcal{H}_a^G \\ - 2\sqrt{g}D_a [M(D^a N + N_b K^{ab}) + N\mathcal{L}M^a - N_b \mathcal{K}^b M^a] \\ + 2\frac{\partial}{\partial \chi} [\sqrt{g}(N\mathcal{L} - N_a \mathcal{K}^a)] + 2\frac{\partial}{\partial t} [M\sqrt{g}(K + \mathcal{K})], \end{aligned} \quad (12)$$

where we have denoted

$$\begin{aligned} \mathcal{H}_\perp^G = -\sqrt{g}[M(R + L^2 - 3L_{ab}L^{ab}) \\ - 2g^{ab}(\partial/\partial \chi - \mathcal{L}_M)L_{ab} - 2D_a D^a M \\ + M(K^2 - K_{ab}K^{ab} - 2\mathcal{K}_a \mathcal{K}^a + 2K\mathcal{K})], \end{aligned} \quad (13a)$$

$$\begin{aligned} \mathcal{H}_a^G = -\sqrt{g}\{D_b[MK^b{}_a - Mg^b{}_a(K + \mathcal{K})] + KD_a M \\ + M\mathcal{K}_a L + (\partial/\partial \chi - \mathcal{L}_M)\mathcal{K}_a\}. \end{aligned} \quad (13b)$$

(Dots represent the time-derivatives $\partial/\partial t$). The advantage of writing the Lagrangian density in the form (12) is that it contains explicitly the Liouville form, with the right coefficient to drop out when performing later on the Legendre transformation. The cofactors \mathcal{H}_\perp^G and \mathcal{H}_a^G of the Lagrange multipliers N and N^a are the super-Hamiltonian and the super-momentum constraints of vacuum gravity, also obtainable as projections of the Einstein tensor, cf. Appendix A:

$$\mathcal{H}_\perp^G = -2M\sqrt{g}n^a n^b \tilde{G}_{ab}, \quad (14a)$$

$$\mathcal{H}_a^G = -2M\sqrt{g}g^b{}_a n^c \tilde{G}_{bc}. \quad (14b)$$

Since the $(s+1)$ -th component of the shift was set to zero, the super-momentum contains only the components corresponding to brane spatial diffeomorphisms.

Now we define the phase space of the brane vacuum gravity as the set of canonical coordinates and canonically conjugated momenta,

$$\{g_A; \pi^A\} := \{g_{ab}, M^a, M; \pi^{ab}, p_a, p\}, \quad (15a)$$

by introducing the notation $g_A = \{g_{ab}, M^a, M\}$ and $\pi^A = \{\pi^{ab}, p_a, p\}$ with the multi-index $A = 1, 2, 3$

as a condensed notation for the gravi-tensorial, gravi-vectorial and gravi-scalar degrees of freedom. The momenta canonically conjugated to the field variables g_{ab} , M^a and M are

$$\pi^{ab} := \frac{\partial \mathcal{L}^G}{\partial \dot{g}_{ab}} = M\sqrt{g} [K^{ab} - (K + \mathcal{K}) g^{ab}] , \quad (16a)$$

$$p_a := \frac{\partial \mathcal{L}^G}{\partial \dot{M}^a} = 2\sqrt{g}\mathcal{K}_a , \quad (16b)$$

$$p := \frac{\partial \mathcal{L}^G}{\partial \dot{M}} = -2\sqrt{g}K . \quad (16c)$$

Inverting Eqs. (16) with respect to the second and normal fundamental forms and the normal fundamental scalar, we obtain

$$K_{ab} = \frac{1}{\sqrt{g}M} \left(\pi_{ab} - \frac{1}{s}g_{ab}\pi \right) - \frac{p}{2s\sqrt{g}}g_{ab} , \quad (17a)$$

$$\mathcal{K}^a = \frac{p^a}{2\sqrt{g}} , \quad (17b)$$

$$\mathcal{K} = \frac{s-1}{2s\sqrt{g}}p - \frac{\pi}{sM\sqrt{g}} . \quad (17c)$$

We insert these formulae into the Lagrangian (11), so that the Einstein-Hilbert action takes the "already Hamiltonian form":

$$\begin{aligned} S^G[g_A, \pi^A; N^a, N] &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\chi \int_{\Sigma_{t\chi}} ds x \\ &\times (\pi^A \dot{g}_A - N \mathcal{H}_\perp^G[g_A, \pi^A] - N^a \mathcal{H}_a^G[g_A, \pi^A]) \\ &- ST_{t=\pm\infty} - ST_{\partial\Sigma_{t\chi}} + ST_{\chi=\pm\infty} . \end{aligned} \quad (18)$$

The extremization of the action (18) with respect to the lapse and shift gives the constraint equations

$$\mathcal{H}_\perp^G = -\frac{\delta S^G}{\delta N} , \quad \mathcal{H}_a^G = -\frac{\delta S^G}{\delta N^a} . \quad (19)$$

In detail, the super-Hamiltonian constraint \mathcal{H}^G and the super-momentum constraint \mathcal{H}_a^G of vacuum gravity, written in terms of the canonical data, are:

$$\begin{aligned} \mathcal{H}_\perp^G[g_A, \pi^A] &= -\sqrt{g}[M(R - L^2 + 3L_{ab}L^{ab}) \\ &- 2g^{ab}(\partial/\partial\chi - \mathcal{L}_M)L_{ab} - 2D_aD^aM] \\ &+ \frac{1}{M\sqrt{g}} \left(\pi_{ab}\pi^{ab} - \frac{1}{s}\pi^2 \right) - \frac{p\pi}{s\sqrt{g}} \\ &+ \frac{M}{2\sqrt{g}} \left(p^a p_a + \frac{s-1}{2s}p^2 \right) , \end{aligned} \quad (20a)$$

$$\begin{aligned} \mathcal{H}_a^G[g_A, \pi^A] &= -2D_b\pi^b_a - (\partial/\partial\chi - \mathcal{L}_M)p_a \\ &+ pD_aM . \end{aligned} \quad (20b)$$

The contributions $ST_{t=\pm\infty}$, $ST_{\partial\Sigma_{t\chi}}$ and $ST_{\chi=\pm\infty}$ in the action (18) denote a collection of surface terms on $t \rightarrow \pm\infty$, $\partial\Sigma_{t\chi}$ and $\chi \rightarrow \pm\infty$, respectively. Their contribution can be compensated by adding surface terms to the action. The contributions $\partial\Sigma_{t\chi}$ and $\chi \rightarrow \pm\infty$ do not

have analogues in the standard ADM decomposition; and they come from partial integrations meant to transform the set of variables (N, N^a) into Lagrange-multipliers in the action. For completeness, we enlist these terms:

$$\begin{aligned} ST_{t=\pm\infty} &= \frac{2}{s} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\chi \int_{\Sigma_{t\chi}} ds x \\ &\times \frac{\partial}{\partial t} \left(\pi + \frac{M}{2}p \right) , \end{aligned} \quad (21a)$$

$$\begin{aligned} ST_{\partial\Sigma_{t\chi}} &= 2 \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\chi \int_{\Sigma_{t\chi}} ds x D_a [N_b \pi^{ab} \\ &- \frac{1}{s}N^a \left(\pi + \frac{M}{2}p \right) - \frac{1}{2}N^b p_b M^a \\ &+ \sqrt{g} (N \mathcal{L} M^a + M D^a N)] , \end{aligned} \quad (21b)$$

$$\begin{aligned} ST_{\chi=\pm\infty} &= 2 \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\chi \int_{\Sigma_{t\chi}} ds x \\ &\times \frac{\partial}{\partial \chi} \left(N \sqrt{g} \mathcal{L} - N^a \frac{p_a}{2} \right) . \end{aligned} \quad (21c)$$

The Hamiltonian of vacuum gravity can be written as the smearing of the super-Hamiltonian and super-momentum constraints with the lapse function and shift vector,

$$H^G[N] = H_\perp^G[N] + H_a^G[N^a] , \quad (22)$$

where

$$H_\perp^G[N] = \int d\chi \int_{\Sigma_{t\chi}} dx^s N(x, \chi) \mathcal{H}_\perp^G(x, \chi) , \quad (23a)$$

$$H_a^G[N^a] = \int d\chi \int_{\Sigma_{t\chi}} dx^s N^a(x, \chi) \mathcal{H}_a^G(x, \chi) . \quad (23b)$$

The Poisson bracket of any two functions $f(x, \chi; g_A, \pi^A)$ and $g(x, \chi; g_A, \pi^A)$ on the phase space is defined as

$$\begin{aligned} \{f(x, \chi), h(x', \chi')\} &= \\ &\int d\chi'' \int_{\Sigma_{t\chi}} ds x'' \frac{\delta f(x, \chi)}{\delta g_A(x'', \chi'')} \frac{\delta h(x', \chi')}{\delta \pi^A(x'', \chi'')} \\ &- \int d\chi'' \int_{\Sigma_{t\chi}} ds x'' \frac{\delta f(x, \chi)}{\delta \pi^A(x'', \chi'')} \frac{\delta h(x', \chi')}{\delta g_A(x'', \chi'')} , \end{aligned} \quad (24a)$$

which provides the following non-vanishing commutation relations for the Poisson brackets of the canonical variables:

$$\{g_A(x, \chi), \pi^{cd}(x', \chi')\} = \delta^c_{(a} \delta^{d)}_{(b} \delta(x, \chi; x', \chi') \quad (25a)$$

$$\{M^a(x, \chi), p_b(x', \chi')\} = \delta^a_b \delta(x, \chi; x', \chi') , \quad (25b)$$

$$\{M(x, \chi), p(x', \chi')\} = \delta(x, \chi; x', \chi') . \quad (25c)$$

or with the condensed notation

$$\{g_A(x, \chi), \pi^B(x', \chi')\} = \delta^B_A \delta(x, \chi; x', \chi') . \quad (26)$$

Here $\delta^B_A = \{\delta^c_{(a} \delta^{d)}_{(b)}, \delta^a_b, 1\}$, $\delta(x, \chi; x', \chi') = \delta(x, x')\delta(\chi, \chi')$ and the Dirac delta distribution behaves

under coordinate transformations as a scalar in its non-primed arguments, and as a scalar density of weight one in its primed ones.

The constraints represent restrictions on the initial data and their Poisson brackets "close" according to the Dirac algebra:

$$\{\mathcal{H}_{\perp}^G(x, \chi), \mathcal{H}_{\perp}^G(x', \chi')\} = g^{ab}(x, \chi)\mathcal{H}_a^G(x, \chi)\delta_{,b}(x, \chi; x', \chi') - (x\chi \leftrightarrow x'\chi') , \quad (27a)$$

$$\{\mathcal{H}_{\perp}^G(x, \chi), \mathcal{H}_a^G(x', \chi')\} = \mathcal{H}_{\perp}^G(x, \chi)\delta_{,a}(x, \chi; x', \chi') + \mathcal{H}_{,a}^G(x, \chi)\delta(x, \chi; x', \chi') , \quad (27b)$$

$$\{\mathcal{H}_a^G(x, \chi), \mathcal{H}_b^G(x', \chi')\} = \mathcal{H}_b^G\delta_{,a}(x, \chi; x', \chi') - (ax\chi \leftrightarrow bx'\chi') . \quad (27c)$$

Time evolution of the canonical data is generated by the Hamiltonian of the system as

$$\dot{g}_A(x, \chi) = \{g_A(x, \chi), H^G[N]\} = \frac{\delta H^G[N]}{\delta \pi^A(x, \chi)} , \quad (28a)$$

$$\dot{\pi}^A(x, \chi) = \{\pi^A(x, \chi), H^G[N]\} = -\frac{\delta H^G[N]}{\delta g_A(x, \chi)} . \quad (28b)$$

Computation gives the dynamical equations for the gravi-tensor, gravi-vector and gravi-scalar degrees of freedom:

$$\dot{g}_{ab} = \frac{2N}{\sqrt{g}} \left[\frac{1}{M} \left(\pi_{ab} - \frac{1}{s}\pi g_{ab} \right) - \frac{1}{2s}pg_{ab} \right] + \mathcal{L}_{\mathbf{N}}g_{ab} , \quad (29a)$$

$$\dot{M}^a = \frac{MN}{\sqrt{g}}p^a + \frac{\partial N^a}{\partial \chi} + \mathcal{L}_{\mathbf{N}}M^a , \quad (29b)$$

$$\dot{M} = \frac{MN}{s\sqrt{g}} \left(\frac{s-1}{2}p - \frac{1}{M}\pi \right) + \mathcal{L}_{\mathbf{N}}M \quad (29c)$$

and

$$\begin{aligned} \dot{\pi}^{ab} = & N\mathcal{S}^{ab} + N\mathcal{V}^{ab} - NM\sqrt{g}\mathcal{L}(L^{ab} - Lg^{ab}) \\ & + \sqrt{g}(D^aD^bN - g^{ab}D^cD_cN - g^{ab}D_cND^cM) \\ & + \sqrt{g}g^{ab}(\partial/\partial\chi - \mathcal{L}_{\mathbf{M}})(N\mathcal{L}) + \mathcal{L}_{\mathbf{N}}\pi^{ab} , \end{aligned} \quad (30a)$$

$$\begin{aligned} \dot{p}_a = & N\mathcal{V}_a - 2\sqrt{g}[L^b{}_aD_bN + D_a(N\mathcal{L})] \\ & + \mathcal{L}_{\mathbf{N}}p_a , \end{aligned} \quad (30b)$$

$$\begin{aligned} \dot{p} = & N\mathcal{S} + N\mathcal{V} - 2\sqrt{g}(L\mathcal{L} + D_aD^aN) \\ & + \mathcal{L}_{\mathbf{N}}p . \end{aligned} \quad (30c)$$

Here \mathcal{S}^{ab} and \mathcal{S} denote the tensorial and scalar projections² of the geodesic spray of the DeWitt super-metric [10]:

² Note that no vectorial projection \mathcal{S}_a is present in \dot{p}_a . However as time-derivatives and index raising do not commute, there will be vectorial kinetic-type terms in \dot{p}^a .

$$\begin{aligned} \mathcal{S}^{ab}(\pi^A, \pi^A) = & -\frac{2}{M\sqrt{g}} \left(\pi^a{}_c\pi^{bc} - \frac{1}{s}\pi\pi^{ab} \right) \\ & + \frac{1}{2M\sqrt{g}} \left(\pi_{cd}\pi^{cd} - \frac{1}{s}\pi^2 \right) g^{ab} \\ & - \frac{M}{2\sqrt{g}}g^{ab} \left(\frac{1}{sM}\pi p - \frac{1}{2}p_c p^c - \frac{s-1}{4s}p^2 \right) \\ & + \frac{1}{\sqrt{g}} \left(\frac{1}{s}p\pi^{ab} + \frac{M}{2}p^a p^b \right) , \end{aligned} \quad (31a)$$

$$\begin{aligned} \mathcal{S}(\pi^A, \pi^A) = & \frac{1}{\sqrt{g}M^2} \left(\pi_{ab}\pi^{ab} - \frac{1}{s}\pi^2 \right) \\ & - \frac{1}{\sqrt{g}} \left(\frac{1}{2}p_a p^a + \frac{s-1}{4s}p^2 \right) , \end{aligned} \quad (31b)$$

while \mathcal{V}^{ab} , \mathcal{V}_a and \mathcal{V} represent the tensorial, vectorial and scalar projections of the force term of the $(s+1)$ -dimensional scalar curvature potential:

$$\begin{aligned} \mathcal{V}^{ab}(g_A) = & -M\sqrt{g}(G^{ab} + 2L^{ac}L^b{}_c - LL^{ab}) \\ & - \frac{1}{2}M\sqrt{g}(3L^{cd}L_{cd} - L^2)g^{ab} \\ & + \sqrt{g}(g^{ac}g^{bd} - g^{ab}g^{cd})(\partial/\partial\chi - \mathcal{L}_{\mathbf{N}})L_{cd} \\ & + \sqrt{g}(D^aD^bM - g^{ab}D^cD_cM) , \end{aligned} \quad (32a)$$

$$\mathcal{V}_a(g_A) = -2\sqrt{g}(D_bL^b{}_a - D_aL) , \quad (32b)$$

$$\mathcal{V}(g_A) = \sqrt{g}(R + L_{ab}L^{ab} - L^2) . \quad (32c)$$

The evolution equations (30) together with the constraints (20) are equivalent with the vacuum Einstein equations in the bulk. Once the constraints are obeyed at some instant of time, the dynamical equations assure that they will continue to be satisfied later on.

IV. HAMILTONIAN DYNAMICS WITH MATTER SOURCES

A. General considerations

The basic scheme of the Hamiltonian formulation does not change if we couple matter fields to gravity. We only have to enlarge the phase space with the canonical variables of the matter sources. The total action describing the system is

$$S = S^G[\tilde{g}_{ab}] + 2\tilde{\kappa}^2 S^M[\tilde{g}_{ab}, \Psi_i] \quad (33)$$

where $\tilde{\kappa}^2$ is the gravitational coupling constant in $s+2$ dimensions and we assume the matter action $S^M[\tilde{g}_{ab}, \Psi_i]$ contains the metric only in non-derivative terms. This assumption is obeyed for all physically relevant matter fields and it assures that (a) the vacuum gravitational momenta (16) remain unchanged in the presence of matter; (b) by performing the Legendre transformation, the

matter contribution to the total Hamiltonian is just minus its contribution to the total Lagrangian.

Extremizing the matter action with respect to the matter fields Ψ_i yields their evolution equations. Extremizing with respect to the metric yields, by definition, the energy-momentum tensor

$$\tilde{T}^{ab} = \frac{2}{\sqrt{-\tilde{g}}} \frac{\delta S^M}{\delta \tilde{g}_{ab}}. \quad (34)$$

We can perform the $(s+1+1)$ -decomposition of the matter Lagrangian density, without specifying it explicitly. For this we write the energy-momentum tensor as

$$\begin{aligned} \tilde{T}^{ab} = & \left(\tilde{T}^{cd} g_c^a g_d^b \right) + \left(\tilde{T}^{cd} n_c n_d \right) n^a n^b \\ & + \left(\tilde{T}^{cd} l_c l_d \right) l^a l^b - 2 \left(\tilde{T}^{cd} g_c^{(a} n_d \right) n^{b)} \\ & + 2 \left(\tilde{T}^{cd} g_c^{(a} l_d \right) l^{b)} - 2 \left(\tilde{T}^{cd} n_c l_d \right) n^{(a} l^{b)}, \end{aligned} \quad (35)$$

and we note that the decomposition Eq. (3) of the metric gives

$$\delta \tilde{g}_{ab} = \delta g_{ab} - 2n_{(a} \delta n_{b)} + 2l_{(a} \delta l_{b)}. \quad (36)$$

The variation of the matter action with respect to the metric (after partial integrations) results in

$$\begin{aligned} \delta \tilde{g} S^M = & \int d^{s+2}x \frac{\delta S^M}{\delta \tilde{g}_{ab}} \delta \tilde{g}_{ab} = \\ & \int d^{s+2}x NM \sqrt{g} \left\{ \frac{1}{2} \left(\tilde{T}^{cd} g_c^a g_d^b \right) \delta g_{ab} \right. \\ & - \left[\tilde{T}^{cd} g_c^a n_d - \left(\tilde{T}^{cd} n_c n_d \right) n^a + \left(\tilde{T}^{cd} n_c l_d \right) l^a \right] \delta n_a \\ & + \left[\tilde{T}^{cd} g_c^a l_d - \left(\tilde{T}^{cd} n_c l_d \right) n^a + \left(\tilde{T}^{cd} l_c l_d \right) l^a \right] \delta l_a \\ & \left. + \left(\tilde{T}^{cd} g_{bc} n_d \right) \delta n^b - \left(\tilde{T}^{cd} g_{bc} l_d \right) \delta l^b \right\}. \end{aligned} \quad (37)$$

We would like to replace the variations δn^a , δl^a , δn_a , δl_a by the variation of our chosen gravitational variables. As $\partial/\partial t$ and $\partial/\partial \chi$ are directions unaffected by the variation of the metric, from Eqs. (1) and (2) we obtain

$$\delta n^a = - \left(\frac{\delta N}{N} n^a + \frac{\delta N^a}{N} \right), \quad (38a)$$

$$\delta l^a = - \left(\frac{\delta M}{M} l^a + \frac{\delta M^a}{M} \right). \quad (38b)$$

As, cf. to Paper I, the dual bases are related as $(dt)_a = n_a/N$ and $(d\chi)_a = l_a/M$, the variation of the co-vectors also arises:

$$\delta n_a = \frac{\delta N}{N} n_a, \quad \delta l_a = \frac{\delta M}{M} l_a, \quad (39)$$

and we obtain the desired formula:

$$\begin{aligned} \delta \tilde{g} S^M [g_{ab}, M^a, M; N^a, N; \Psi] = & \int dt \int d\chi \int_{\Sigma_{t\chi}} ds x \sqrt{g} \left\{ \frac{NM}{2} \left(\tilde{T}^{cd} g_c^a g_d^b \right) \delta g_{ab} \right. \\ & + N \left(\tilde{T}^{cd} l_c l_d \right) \delta M + N \left(\tilde{T}^{cd} g_c^b l_d \right) g_{ab} \delta M^a \\ & \left. - M \left(\tilde{T}^{cd} n_c n_d \right) \delta N - M \left(\tilde{T}^{cd} g_c^b n_d \right) g_{ab} \delta N^a \right\} \end{aligned} \quad (40)$$

(Due to the non-derivative coupling of matter to gravity there are no momenta dependencies.) The result (40) shows that extremizing the total action with respect to the lapse function N and the shift vector N^a (similarly to the prescription (19)) gives the super-Hamiltonian and super-momentum contribution of the matter fields

$$\mathcal{H}_\perp^M = 2\tilde{\kappa}^2 M \sqrt{g} \left(\tilde{T}^{cd} n_c n_d \right), \quad (41a)$$

$$\mathcal{H}_a^M = 2\tilde{\kappa}^2 M \sqrt{g} \left(\tilde{T}^{cd} g_c^b n_d \right) g_{ab}. \quad (41b)$$

The super-Hamiltonian and super-momentum constraints of the total system can be written as

$$\mathcal{H}_\perp = \mathcal{H}_\perp^G + \mathcal{H}_\perp^M \approx 0, \quad (42a)$$

$$\mathcal{H}_a = \mathcal{H}_a^G + \mathcal{H}_a^M \approx 0, \quad (42b)$$

with the vacuum and matter contributions given by Eqs. (20) and Eqs. (41). Here \approx denotes weak equality (holding on the constraint surface in the phase space).

In what follows, we discuss the canonical equations in the presence of matter. Due to the non-derivative coupling, Eqs. (29) remain valid in the presence of matter (since $\delta S^M/\delta \pi^A = 0$). However the evolution of the momenta receive additional contributions. Due to remark (b), the matter contributions to the left hand side of Eqs. (30) can be found by extremizing the action with respect to the dynamical variables g_{ab} , M^a and M :

$$\frac{\delta S^M}{\delta g_{ab}} = \frac{NM}{2} \sqrt{g} \left(\tilde{T}^{cd} g_c^a g_d^b \right), \quad (43a)$$

$$\frac{\delta S^M}{\delta M^a} = N \sqrt{g} \left(\tilde{T}^{cd} g_{ac} l_d \right), \quad (43b)$$

$$\frac{\delta S^M}{\delta M} = N \sqrt{g} \left(\tilde{T}^{cd} l_c l_d \right). \quad (43c)$$

The dynamical equations for π^A with the contributions (43) take the form

$$\begin{aligned} \dot{\pi}^{ab} = & N S^{ab} + N \mathcal{V}^{ab} - NM \sqrt{g} \mathcal{L} (L^{ab} - L g^{ab}) \\ & + \sqrt{g} (D^a D^b N - g^{ab} D^c D_c N - g^{ab} D_c N D^c M) \\ & + \sqrt{g} g^{ab} (\partial/\partial \chi - \mathcal{L}_M) (N \mathcal{L}) + \mathcal{L}_N \pi^{ab} \\ & + \tilde{\kappa}^2 NM \sqrt{g} \left(\tilde{T}^{cd} g_c^a g_d^b \right), \end{aligned} \quad (44a)$$

$$\begin{aligned} \dot{p}_a = & N \mathcal{V}_a N \mathcal{V}_a - 2\sqrt{g} [L^b_a D_b N + D_a (N \mathcal{L})] \\ & + \mathcal{L}_N p_a + 2\tilde{\kappa}^2 N \sqrt{g} \left(\tilde{T}^{bc} g_{ab} l_c \right), \end{aligned} \quad (44b)$$

$$\begin{aligned} \dot{p} = & N \mathcal{S} + N \mathcal{V} - 2\sqrt{g} (L \mathcal{L} + D_a D^a N) \\ & + \mathcal{L}_N p + 2\tilde{\kappa}^2 N \sqrt{g} \left(\tilde{T}^{ab} l_a l_b \right). \end{aligned} \quad (44c)$$

These formulae are valid for any matter source coupled non-derivatively to the $(s+2)$ -geometry. The matter contributions can be further specified, once the energy-momentum tensor (or equivalently, the matter Lagrangian) is known.

B. Brane-world scenario

In the brane-world scenarios the stress-energy tensor is decomposed as

$$\tilde{T}_{ab} = \tilde{\Pi}_{ab} + [-\lambda^{(s+1)}g_{ab} + {}^{(s+1)}T_{ab}]\delta(\chi), \quad (45)$$

where the regular part $\tilde{\Pi}_{ab}$ represents the non-standard model bulk sources, while the distributional term contains the brane tension λ and the energy-momentum tensor of standard model matter field localized on the brane. First we decompose the bulk energy momentum $\tilde{\Pi}_{ab}$ with respect to the off-brane normal l^a as:

$$\frac{s-1}{s}\tilde{\Pi}_{ab} = {}^{(s+1)}\mathcal{P}_{ab} + 2l_{(a}{}^{(s+1)}\mathfrak{P}_{b)} + l_a l_b \mathfrak{P}. \quad (46)$$

Then we decompose ${}^{(s+1)}\mathcal{P}_{ab}$ and ${}^{(s+1)}\mathfrak{P}_b$ further with respect to the time-like normal n^a as:

$${}^{(s+1)}\mathcal{P}_{ab} = \mathcal{P}_{ab} + 2n_{(a}\mathcal{P}_{b)} + n_a n_b \mathcal{P}, \quad (47a)$$

$${}^{(s+1)}\mathfrak{P}_a = \mathcal{P}_a + \mathcal{P} n_a, \quad (47b)$$

where $\mathcal{P}_{ab}n^b = \mathcal{P}_a n^a = 0 = \mathcal{P}_a n^a$. (Note that $g^{ab}\mathcal{P}_{ab} \neq \mathcal{P}$, unless ${}^{(s+1)}\mathcal{P}_{ab}$ happens to be traceless.)

The brane contribution can be algebraically decomposed as

$${}^{(s+1)}T_{ab} = \rho n_a n_b + P g_{ab} + \Pi_{ab} + 2n_{(a}Q_{b)} \quad (48)$$

with respect to the 4-velocity n^a of the fluid, and in terms of the energy density ρ , isotropic pressure P , anisotropic stresses Π_{ab} and the energy transport (heat flow) Q_a (here $g^{ab}\Pi_{ab} = n^a\Pi_{ab} = n^aQ_a = 0$).

Choosing the 4-velocity of the fluid as normal to the spatial slices does not restrict the arbitrariness of the foliation. Indeed, the foliation is given by the form $(dt)_a = n_a/N$, while the normal vector field is $n^a = \tilde{g}^{ab}(dt)_b/N$, which involves the $(s+2)$ -metric, and as such, the lapse and shift. The arbitrariness of the lapse function and shift vector assures that one can still choose various foliations once n^a is fixed by the chosen reference fluid. The time parameter t_{ref} defined by the fluid as $n^a = (\partial/\partial t_{ref})^a$ is different from the time t defined by the chosen foliation. Therefore while we associate t_{ref} with the cosmological time, we still have the freedom of evolving the system with respect to any conveniently chosen time parameter t (like the conformal time). Restricting $N = 1$ and $N^a = 0$ leads to the identification $t = t_{ref}$.

The above conditions with Eqs. (41) and Eqs. (43) give

$$\mathcal{H}_{\perp}^M[g_A; \mathcal{P}, \rho] = 2\tilde{\kappa}^2 M \sqrt{g} \left[\frac{s}{s-1} \mathcal{P} + (\rho + \lambda) \delta(\chi) \right], \quad (49a)$$

$$\mathcal{H}_a^M[g_A; \mathcal{P}_a, Q_a] = -2\tilde{\kappa}^2 M \sqrt{g} \left[\frac{s}{s-1} \mathcal{P}_a + Q_a \delta(\chi) \right], \quad (49b)$$

and

$$\begin{aligned} \frac{\delta S^M}{\delta g_{ab}} &= \frac{NM}{2} \sqrt{g} \frac{s}{s-1} \mathcal{P}^{ab} \\ &+ \frac{NM}{2} \sqrt{g} [\Pi^{ab} + (P - \lambda) g^{ab}] \delta(\chi), \end{aligned} \quad (50a)$$

$$\frac{\delta S^M}{\delta M^a} = N \sqrt{g} \frac{s}{s-1} \mathcal{P}_a, \quad (50b)$$

$$\frac{\delta S^M}{\delta M} = N \sqrt{g} \frac{s}{s-1} \mathfrak{P}. \quad (50c)$$

The full brane-world geometrodynamics in the presence of matter is then given by the equations (29) for \dot{g}_A and the sum of the right hand sides of Eqs. (30) and (50) for $\dot{\pi}^A$:

$$\begin{aligned} \dot{\pi}^{ab} &= N \mathcal{S}^{ab} + N \mathcal{V}^{ab} - NM \sqrt{g} \mathcal{L}(L^{ab} - L g^{ab}) \\ &+ \sqrt{g} (D^a D^b N - g^{ab} D^c D_c N - g^{ab} D_c N D^c M) \\ &+ \sqrt{g} g^{ab} (\partial/\partial \chi - \mathcal{L}_M)(N \mathcal{L}) + \mathcal{L}_N \pi^{ab} \\ &+ \tilde{\kappa}^2 NM \sqrt{g} \left\{ \frac{s}{s-1} \mathcal{P}^{ab} \right. \\ &\left. + [\Pi^{ab} + (P - \lambda) g^{ab}] \delta(\chi) \right\}, \end{aligned} \quad (51a)$$

$$\begin{aligned} \dot{p}_a &= N \mathcal{V}_a - 2\sqrt{g} [L^b_a D_b N + D_a (N \mathcal{L})] \\ &+ \mathcal{L}_N p_a + \tilde{\kappa}^2 N \sqrt{g} \frac{2s}{s-1} \mathcal{P}_a, \end{aligned} \quad (51b)$$

$$\begin{aligned} \dot{p} &= N \mathcal{S} + N \mathcal{V} - 2\sqrt{g} (L \mathcal{L} + D_a D^a N) \\ &+ \mathcal{L}_N p + \tilde{\kappa}^2 N \sqrt{g} \frac{2s}{s-1} \mathfrak{P}. \end{aligned} \quad (51c)$$

The constraints (42a) and (42b) take the form

$$\begin{aligned} 0 \approx \mathcal{H}_{\perp}[g_A, \pi^A; \rho, \mathcal{P}] &= -\sqrt{g} M (R - L^2 + 3L_{ab} L^{ab}) \\ &- 2\sqrt{g} [g^{ab} (\partial/\partial \chi - \mathcal{L}_M) L_{ab} - D_a D^a M] \\ &+ \frac{1}{M \sqrt{g}} \left(\pi_{ab} \pi^{ab} - \frac{1}{s} \pi^2 \right) - \frac{p \pi}{s \sqrt{g}} \\ &+ \frac{M}{2\sqrt{g}} \left(p^a p_a + \frac{s-1}{2s} p^2 \right) \\ &+ 2\tilde{\kappa}^2 M \sqrt{g} \left[\frac{s}{s-1} \mathcal{P} + (\rho + \lambda) \delta(\chi) \right], \end{aligned} \quad (52a)$$

$$\begin{aligned} 0 \approx \mathcal{H}_a[g_A, \pi^A; \mathcal{P}_a, Q_a] &= -2D_b \pi^b_a - (\partial/\partial \chi - \mathcal{L}_M) p_a + p D_a M \\ &- 2\tilde{\kappa}^2 M \sqrt{g} \left[\frac{s}{s-1} \mathcal{P}_a + Q_a \delta(\chi) \right]. \end{aligned} \quad (52b)$$

The dynamical equations (29) and (51) and the constraints (52) completely determine the time evolution of the geometry and the matter fields on the brane in brane-world scenarios. The LSDI matching condition follows from these equations, as we will show it in the next section.

The δ -function type distributional sources in the evolution equations (51a) need some further interpretation. Such contributions also appear in the dynamics of K_{ab} and \mathcal{K} , as derived in Paper I. These contributions indicate the singular behavior of $\dot{\pi}^{ab}$. This is however, not surprising. The canonical equations and constraints are equivalent with the $(s+2)$ -dimensional Einstein equations. If the sources of the latter are singular (in the present case across the brane), the Riemann (and Einstein) tensors are also singular, and certain singularities will be carried over in the canonical equations. Traditionally (for example in the derivation of the effective Einstein equation [7]) the coefficients of the δ -functions are interpreted as contributions present on the brane, but not in the bulk regions.

V. REGULARIZATION ACROSS THE BRANE

The brane contains δ -function type distributional sources, which in turn appear in both constraints (52) and in the dynamical equation (51a). For these equations we apply the following regularization procedure. First we consider a domain of finite thickness $(-\chi_0, \chi_0)$ enclosing the brane at $\chi = 0$ and we integrate the above-derived equations across its width. As a consequence, the Dirac distribution $\delta(\chi)$ disappears. More precisely, according to [8], for any $\mathcal{H}(l)$ the relation

$$\int_{-l_0}^{l_0} dl \delta(l) \mathcal{H}(l) = \mathcal{H}(0) \quad (53)$$

holds. The integration across the brane is carried out here over a normal coordinate l , defined as $\partial/\partial l = \mathbf{1}$ (see [8]). Should we employ the coordinate χ , defined as $\partial/\partial \chi = M\mathbf{1} + \mathbf{M}$ for integration across the brane (thus $dl/d\chi = M$), we obtain

$$\int_{-\chi_0}^{\chi_0} d\chi \delta(\chi) \mathcal{H}(\chi) = \frac{\mathcal{H}(0)}{M} . \quad (54)$$

Secondly, the primitive function of the integral of any total χ -derivative term evaluated at the left and right domain boundaries $-\chi_0$ and χ_0 give the so-called jump of the respective quantities. Thus, any quantity \mathcal{G} appearing as $\partial\mathcal{G}/\partial\chi$ in the respective equation leads to its jump across the brane $\Delta\mathcal{G}$, when the integration is carried out. Finally, if the value of χ_0 is small, any other smooth function of χ_0 can be regarded as a constant, such that its integral will be proportional to the width $2\chi_0$. When we take the thin brane limit $\chi_0 \rightarrow 0$, this procedure drops all such terms, and what remains are only the terms originally multiplying δ -functions and the jumps arising from

the total χ -derivatives. If no such terms are present in any of the equations derived in the preceding sections, we obtain identities (of $0 = 0$ type). Therefore non-trivial information arises only from the equations with total χ -derivatives and/or δ -functions.

We can regain the junction condition for the embedding of the brane by integrating the constraints and those dynamical equations which contain derivatives with respect to χ , the coordinate running in the off-brane direction.

As Eqs. (20a) and (20b) show, the vacuum constraints contain the χ -derivatives of L_{ab} and p_a . The integration of the super-Hamiltonian constraint (42a) provides

$$\sqrt{g} g^{ab} \Delta L_{ab} = -\tilde{\kappa}^2 \lim_{\chi_0 \rightarrow 0} \int_{-\chi_0}^{\chi_0} d\chi M \sqrt{g} \left(\tilde{T}^{cd} n_c n_d \right) . \quad (55)$$

By considering the brane world scenario, the integration of Eq. (52a) simply gives the trace of the junction condition (53) of Paper I:

$$\Delta L = -\tilde{\kappa}^2 (\rho + \lambda) , \quad (56)$$

$$\text{since } \int_{-\chi_0}^{\chi_0} d\chi f(\chi) \delta(\chi) = f(0)/M .$$

When we integrate the supermomentum constraint (42b), we obtain

$$\Delta p_a = 2\tilde{\kappa}^2 \lim_{\chi_0 \rightarrow 0} \int_{-\chi_0}^{\chi_0} d\chi M \sqrt{g} \left(\tilde{T}^{cd} g_c^b n_d \right) g_{ab} , \quad (57)$$

since the integral of the finite terms in the momentum constraint vanishes as $\chi_0 \rightarrow 0$. For the matter fields on the brane specified in Eq. (49b), the integration of the momentum constraint (52b) gives

$$\Delta p_a = -2\tilde{\kappa}^2 \sqrt{g} Q_a , \quad (58)$$

which is the vectorial projection (B12) of the Lanczos equation, rewritten in terms of momenta.

By integrating the dynamical equation (44a) over χ , we obtain

$$\begin{aligned} & \sqrt{g} [\Delta L^{ab} - \Delta(L - \mathcal{L}) g^{ab}] \\ &= -\tilde{\kappa}^2 \lim_{\chi_0 \rightarrow 0} \int_{-\chi_0}^{\chi_0} d\chi M \sqrt{g} \tilde{T}^{cd} g_c^a g_d^b . \end{aligned} \quad (59)$$

whereas the integration of the dynamical equation (51a) leads to the expression

$$\Delta L_{ab} - \Delta(L - \mathcal{L}) g_{ab} = -\tilde{\kappa}^2 [\Pi_{ab} + (P - \lambda) g_{ab}] . \quad (60)$$

After inserting Eq. (56) in the trace of this result, we get

$$\Delta \mathcal{L} = \tilde{\kappa}^2 \frac{(1-s)\rho - sP + \lambda}{s} , \quad (61)$$

which is the scalar projection (B8) of the Lanczos equation.

The substitution of Eq. (61) into Eq. (60) gives

$$\Delta L_{ab} = -\tilde{\kappa}^2 \left(\Pi_{ab} + \frac{\rho + \lambda}{s} g_{ab} \right) , \quad (62)$$

Then we have obtained the tensorial projection (B6) of the Lanczos equation. This means the dynamical system (51) with the constraints (52) imply the usual LSDI junction conditions for the brane.

By imposing Z_2 symmetry in the bulk across the brane (which implies $\Delta L_{ab} = 2L_{ab}$, $\Delta p_a = 2p_a$, and $\Delta \mathcal{L} = 2\mathcal{L}$), we can express the components of the extrinsic curvature associated with the brane normal in terms of the matter field variables:

$$L_{ab} = -\frac{\tilde{\kappa}^2}{2} \left(\Pi_{ab} + \frac{\rho + \lambda}{s} g_{ab} \right), \quad (63a)$$

$$p_a = -\tilde{\kappa}^2 \sqrt{g} Q_a, \quad (63b)$$

$$\mathcal{L} = \tilde{\kappa}^2 \frac{(1-s)\rho - sP + \lambda}{2s}. \quad (63c)$$

As a simple application, we give the dynamical equation of the heat flow. Eq. (63b) implies

$$\tilde{\kappa}^2 \sqrt{g} \dot{Q}_a = \frac{p_a}{2} \dot{g}^{bc} \dot{g}_{bc} - \dot{p}_a. \quad (64)$$

Here both \dot{g}_{bc} and \dot{p}_a are known as Eqs. (29a) and (51b). We obtain:

$$\begin{aligned} \tilde{\kappa}^2 \sqrt{g} \dot{Q}_a &= p_a \left(-\frac{N}{2\sqrt{g}} p + D_b N^b \right) \\ &\quad - N \mathcal{V}_a + 2\sqrt{g} [L^b_a D_b N + D_a (N \mathcal{L})] \\ &\quad - \mathcal{L}_N p_a - \tilde{\kappa}^2 N \sqrt{g} \frac{2s}{s-1} P_a. \end{aligned} \quad (65)$$

This is the equation of heat flow expressed in terms of canonical data.

VI. CONCLUDING REMARKS

We have derived the Hamiltonian dynamics of the $(s+2)$ -dimensional gravitation in terms of variables adapted to the existence of the preferred s -dimensional hypersurface. The canonical (gravi-tensorial, gravi-vectorial and gravi-scalar) metric variables g_{ab} , M^a and M have canonically conjugated momenta π^{ab} , p_a and p , related to the extrinsic curvatures associated to the temporal normal n^a . We have given the evolution equations for the canonical data and also the super-Hamiltonian and super-momentum constraints, all derived from an action principle.

Some of these equations contain δ -function type contributions, due to the singular source terms on the brane. These terms can be dropped, when we monitor gravitational dynamics in the bulk, and kept on the brane.

The regularization of these equations across the brane yields the projections of the Lanczos equation, written in terms of canonical data.

In the original covariant formulation of brane-world dynamics [7] the effective Einstein equation is obtained by expressing the terms quadratic in the extrinsic curvatures with matter variables. This is achieved by employing the Lanczos equation. In the present formalism, the role of these extrinsic curvatures are taken by L_{ab} , $\mathcal{L}_a = \mathcal{K}_a$ and \mathcal{L} , all functions of the canonical variables. The projections of the Lanczos equation derived in this paper can also be employed to eliminate these geometrical variables in terms of matter variables in the canonical equations.

As a simple application we have derived the equation of heat flow in terms of canonical data.

The importance of the presented formalism relies in its possible application in the initial-value problem in brane-worlds and in the prospect of canonical quantization of brane-world gravity.

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APPENDIX A: THE $s+1+1$ -DECOMPOSITION OF ENERGY-MOMENTUM AND THE EINSTEIN TENSORS

In Appendix D of Paper **I** we have given the complete set of the decompositions of the Riemann-, Ricci- and Einstein tensors. Due to a typo, from the decomposition of the curvature scalar, Eq. (D3) of Paper **I** the term $2(\mathcal{K}^2 - \mathcal{L}^2)$ was omitted, which cancels out the corresponding terms in the projections $g_a^c g_b^d \tilde{G}_{cd}$, $n^a n^b \tilde{G}_{ab}$ and $l^a l^b \tilde{G}_{ab}$. Therefore the formulae (D3)-(D4c) of Paper **I** correctly read:

$$\begin{aligned} \tilde{R} &= \tilde{R} = R - 3K_{ab}K^{ab} + K^2 + 2[(K + \mathcal{K})\mathcal{K} + \mathcal{K}_a\mathcal{K}^a + g^{ab}\mathcal{L}_n K_{ab} + \mathcal{L}_n \mathcal{K}] \\ &\quad + 3L_{ab}L^{ab} - L^2 + 2[(L - \mathcal{L})\mathcal{L} - g^{ab}\mathcal{L}_1 L_{ab} + \mathcal{L}_1 \mathcal{L}] \\ &\quad - 2[N^{-1}D_a D^a N + M^{-1}D_a D^a M + (NM)^{-1}D_a N D^a M] \end{aligned} \quad (A1)$$

and

$$\begin{aligned}
g_a^c g_b^d \tilde{G}_{cd} &= G_{ab} - 2K_{ac}K_b^c + (K + \mathcal{K})K_{ab} - 2\mathcal{K}_a\mathcal{K}_b + \mathcal{L}_n K_{ab} - N^{-1}D_b D_a N \\
&+ \left[\frac{1}{2}(3K_{cd}K^{cd} - K^2) - (K + \mathcal{K})\mathcal{K} - \mathcal{K}_c\mathcal{K}^c - g^{cd}\mathcal{L}_n K_{cd} - \mathcal{L}_n \mathcal{K} + N^{-1}D_c D^c N \right] g_{ab} \\
&+ 2L_{ac}L_b^c - (L - \mathcal{L})L_{ab} - \mathcal{L}_1 L_{ab} - M^{-1}D_b D_a M \\
&- \left[\frac{1}{2}(3L_{cd}L^{cd} - L^2) + \mathcal{L}(L - \mathcal{L}) - g^{cd}\mathcal{L}_1 L_{cd} - \mathcal{L}_1 \mathcal{L} - M^{-1}D_c D^c M \right] g_{ab} \\
&+ (NM)^{-1}D_c N D^c M , \tag{A2a}
\end{aligned}$$

$$g_a^c n^d \tilde{G}_{cd} = D_c K_a^c - D_a (K + \mathcal{K}) + \mathcal{K}_a L + \mathcal{L}_1 \mathcal{K}_a + M^{-1}K_a^c D_c M - M^{-1}\mathcal{K} D_a M , \tag{A2b}$$

$$g_a^c l^d \tilde{G}_{cd} = D_c L_a^c - D_a (L - \mathcal{L}) + \mathcal{K}_a K + \mathcal{L}_n K_a + N^{-1}L_a^c D_c N + N^{-1}\mathcal{L} D_a N , \tag{A2c}$$

$$n^a n^b \tilde{G}_{ab} = \frac{1}{2}(R - K_{ab}K^{ab} + K^2 + 3L_{ab}L^{ab} - L^2) + \mathcal{K}K - \mathcal{K}_a\mathcal{K}^a - g^{ab}\mathcal{L}_1 L_{ab} - M^{-1}D_a D^a M , \tag{A2d}$$

$$l^a l^b \tilde{G}_{ab} = -\frac{1}{2}(R + L_{ab}L^{ab} - L^2 - 3K_{ab}K^{ab} + K^2) - \mathcal{L}L + \mathcal{K}_a\mathcal{K}^a - g^{ab}\mathcal{L}_n K_{ab} + N^{-1}D_a D^a N , \tag{A2e}$$

$$n^a l^b \tilde{G}_{ab} = D_a \mathcal{K}^a - g^{ab}\mathcal{L}_1 K_{ab} + K_{ab}L^{ab} + \mathcal{K}L + M^{-1}\mathcal{K}^a D_a M . \tag{A2f}$$

The last three equations agree with Eqs. (D2d)-(D2f) of Paper **I**. From among them (A2d) correctly yields the Hamiltonian constraint, Eq. (13a).

APPENDIX B: THE DERIVATION OF THE LANCZOS EQUATION

We have demonstrated in Section V. that integration of the dynamical equations and of the constraints across the brane gives the projections of the Lanczos equation in the momentum phase space. In the velocity phase space these projections were obtained directly by projections of the full Lanczos equation, as Eqs. (53), (54), and (55) of Paper **I** (these are however valid only for $s = 3$). Equivalently, we can integrate the dynamical equations (67a), (67c) and the projection (D2e) of the Ricci tensor (i.e., the diffeomorphism constraint given in terms of velocity instead of momentum) of Paper **I** in order to obtain in full generality (for generic s) the projections of the Lanczos equation in the velocity phase-space. With this we fully establish the commutativity of the variation principle and the geometrical decomposition of the quantities, to be discussed in more detail in Appendix C.

In order to carry out this program, we apply the regularization procedure described in Section V. for the evolution equations of K_{ab} and \mathcal{K} , derived in Paper **I**, which contain the following projections of \tilde{R}_{ab} .

$$g_a^c g_b^d \tilde{R}_{cd} = \tilde{\kappa}^2 \left[\frac{2\tilde{\Lambda} + (\rho + \lambda)\delta(\chi)}{s} g_{ab} + \Pi_{ab}\delta(\chi) \right] \tag{B1}$$

$$l^a l^b \tilde{R}_{ab} = \tilde{\kappa}^2 \left[\frac{2\tilde{\Lambda} + [\rho - sp + (s + 1)\lambda]\delta(\chi)}{s} \right] , \tag{B2}$$

The first and third of the Eqs. (67) of Paper **I** can be

rewritten conveniently as

$$\begin{aligned}
\frac{\partial}{\partial t} K_{ab} &= \tilde{\kappa}^2 N \left[\frac{(\rho + \lambda)}{s} g_{ab} + \Pi_{ab} \right] \delta(\chi) \\
&+ \frac{\partial}{\partial \chi} \left(\frac{N}{M} L_{ab} \right) + A_{ab} , \tag{B3a}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t} \mathcal{K} &= \tilde{\kappa}^2 N \frac{[\rho - sp + (s + 1)\lambda]}{s} \delta(\chi) \\
&+ \frac{\partial}{\partial \chi} \left[\frac{N}{M} (L - \mathcal{L}) \right] + A , \tag{B3b}
\end{aligned}$$

where in A_{ab} and A we have collected only *finite* terms:

$$\begin{aligned}
A_{ab} &= N \left[\frac{2}{s} \tilde{\kappa}^2 \tilde{\Lambda} g_{ab} - R_{ab} + L_{ab} (L - \mathcal{L}) - 2L_{ac}L_b^c \right. \\
&- K_{ab} (K + \mathcal{K}) + 2K_{ac}K_b^c + 2\mathcal{K}_a\mathcal{K}_b \\
&\left. + \frac{1}{M} (D_b D_a M - M^c D_c L_{ab} - 2L_{c(a} D_{b)} M^c) \right] \\
&+ D_b D_a N + N^c D_c K_{ab} + 2K_{c(a} D_{b)} N^c \\
&- L_{ab} \frac{\partial}{\partial \chi} \left(\frac{N}{M} \right) , \tag{B4a}
\end{aligned}$$

$$\begin{aligned}
A &= N \left\{ \frac{2}{s} \tilde{\kappa}^2 \tilde{\Lambda} - L_{ab} L^{ab} + \mathcal{L}^2 - 2\mathcal{K}_a \mathcal{K}^a - \mathcal{K}(K + \mathcal{K}) \right. \\
&- \frac{1}{M} M^a D_a (L - \mathcal{L}) + \frac{D_a D^a M}{M} \left. \right\} \\
&+ \frac{D^a M}{M} D_a N + N^a D_a \mathcal{K} \\
&- (L - \mathcal{L}) \frac{\partial}{\partial \chi} \left(\frac{N}{M} \right) . \tag{B4b}
\end{aligned}$$

The integration across a finite coordinate distance containing the brane (for example from $-\chi_0$ to χ_0) of Eqs. (B3) and the subsequent limit $\chi_0 \rightarrow 0$ gives

$$-\tilde{\kappa}^2 \frac{N}{M} \left[\frac{(\rho + \lambda)}{s} g_{ab} + \Pi_{ab} \right] = \Delta \left(\frac{N}{M} L_{ab} \right) , \quad (\text{B5a})$$

$$-\tilde{\kappa}^2 \frac{N}{M} \left[\frac{\rho - sp + (s+1)\lambda}{s} \right] = \Delta \left[\frac{N}{M} (L - \mathcal{L}) \right] \quad (\text{B5b})$$

(In deriving Eqs. (B5) we have employed that time-derivatives and integration over χ commute.) With the remark that neither M , nor N are discontinuous across the brane, but the extrinsic curvature L_{ab} is (as it depends on the brane embedding into the bulk on each side), from Eq. (B5a) we obtain the jump of L_{ab} as

$$\Delta L_{ab} = -\tilde{\kappa}^2 \left[\frac{(\rho + \lambda)}{s} g_{ab} + \Pi_{ab} \right] . \quad (\text{B6})$$

This is the *tensorial projection of the Lanczos equation*, Eq. (53) of Paper **I**, valid for generic s . The trace of the right hand side of this equation is

$$\Delta L = -\tilde{\kappa}^2 (\rho + \lambda) . \quad (\text{B7})$$

With this, Eq. (B5b) implies

$$\Delta \mathcal{L} = \tilde{\kappa}^2 \frac{(1-s)\rho - sp + \lambda}{s} , \quad (\text{B8})$$

which is the *scalar projection of the Lanczos equation*, Eq. (55) of Paper **I**, valid for generic s .

The vectorial projection of the Lanczos equation does not emerge in a similar way. Indeed, the remaining evolution equation for \mathcal{K}_a [the second Eq. (67) of Paper **I**, is

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{K}_a &= N \left[-D^b L_{ab} + D_a (L - \mathcal{L}) - K \mathcal{K}_a \right] + N^b D_b \mathcal{K}^a \\ &\quad - (L_a^b + \mathcal{L} \delta_a^b) D_b N + \mathcal{K}_b D_a N^b . \end{aligned} \quad (\text{B9})$$

As all terms are regular, integration over the range $(-\chi_0, \chi_0)$ and the limit $\chi_0 \rightarrow 0$ would give nothing but the identity $0 = 0$.

The vectorial projection of the Lanczos equation can be instead obtained from Eq. (D2e) of Paper **I**, which contains the projection $g_a^c n^d \tilde{R}_{cd}$, given by the bulk Einstein equation:

$$g_a^c n^d \tilde{R}_{cd} = \tilde{\kappa}^2 \left[g_a^c n^d \tilde{\Pi}_{cd} - Q_a \delta(\chi) \right] .$$

Written in terms of χ -derivatives, Eq. (D2e) of Paper **I** becomes

$$0 = \tilde{\kappa}^2 Q_a \delta(\chi) + \frac{1}{M} \frac{\partial}{\partial \chi} \mathcal{K}_a + A_a \quad (\text{B10})$$

$$\begin{aligned} A_a &= -\tilde{\kappa}^2 g_a^c n^d \tilde{\Pi}_{cd} + D_c K_a^c - D_a (K + \mathcal{K}) + \mathcal{K}_a L \\ &\quad + \frac{1}{M} (K_a^i D_i M - \mathcal{K} D_a M + \mathcal{K}^c D_c M_a - M^c D_c \mathcal{K}_a) , \end{aligned} \quad (\text{B11})$$

where A_a represents the collection of finite terms. Integration over an infinitesimal range $(-\chi_0, \chi_0)$ and taking the limit $\chi_0 \rightarrow 0$ gives

$$\Delta \mathcal{K}_a = -\tilde{\kappa}^2 Q_a . \quad (\text{B12})$$

This is the *vectorial projection of the Lanczos equation*, Eq. (54) of Paper **I**.

APPENDIX C: DYNAMICS REGAINED FROM THE $(s+1)+1$ ADM DECOMPOSITION

In this Appendix we regain the full $(s+1+1)$ -breakup of the $(s+2)$ -dimensional equations by a step-by-step method, by further splitting of the $((s+1)+1)$ -dimensional ADM decomposition of the bulk. First we foliate the $(s+2)$ -dimensional space-time \mathcal{B} with the $(s+1)$ -spaces \mathcal{S}_t . Then we derive the equations of motion together with the constraints from the corresponding variation principle. Finally we embed the s -spaces $\Sigma_{t\chi}$ in the hypersurfaces \mathcal{S}_t and perform a second split-up with respect to the extra dimension of the geometrodynamics.

We begin by briefly presenting the standard $(s+1)+1$ ADM decomposition of the vacuum Lagrangian. This yields:

$$\mathcal{L}^G = N \sqrt{\tilde{g}} (\tilde{R} - \tilde{K}^2 + \tilde{K}_{ab} \tilde{K}^{ab}) - 2 \tilde{\nabla}_a [N \sqrt{\tilde{g}} (\alpha^a - \tilde{K} n^a)] , \quad (\text{C1})$$

with the dynamical variables \tilde{g}_{ab} and \tilde{K}_{ab} , representing the first and second fundamental forms of \mathcal{S}_t , respectively. Here \tilde{R} is the intrinsic curvature scalar of \mathcal{S}_t and $\tilde{K} = \tilde{g}^{ab} \tilde{K}_{ab}$ is the trace of \tilde{K}_{ab} . This equation can be rewritten in the form

$$\begin{aligned} \mathcal{L}^G &= \sqrt{\tilde{g}} \left(\tilde{\mathcal{L}}_t \tilde{K} + \tilde{g}^{ab} \tilde{\mathcal{L}}_t \tilde{K}_{ab} \right) + N \tilde{\mathcal{H}}_\perp^G \\ &\quad + N_a \tilde{\mathcal{H}}_a^G - 2 \sqrt{\tilde{g}} \tilde{D}_a (\tilde{D}^a N + N_b \tilde{K}^{ab}) \end{aligned} \quad (\text{C2})$$

containing the pure spatial projections $\tilde{\mathcal{L}}_t \tilde{K}_{ab} = \tilde{g}^c_a \tilde{g}^d_b \tilde{\mathcal{L}}_t \tilde{K}_{cd}$ and $\tilde{D}_a \tilde{K}_{bc} = \tilde{g}^d_a \tilde{g}^e_b \tilde{g}^f_c \tilde{\nabla}_d \tilde{K}_{ef}$ of the $(s+2)$ -dimensional Lie-derivative $\tilde{\mathcal{L}}_t \tilde{K}_{cd}$ and covariant derivative $\tilde{\nabla}_d \tilde{K}_{ef}$, respectively. We can also identify the cofactors of the Lagrange multipliers N and N^a as the super-Hamiltonian and super-momentum constraints

$$\tilde{\mathcal{H}}_\perp^G = -2 \sqrt{\tilde{g}} n^a n^b \tilde{G}_{ab} = \sqrt{\tilde{g}} (\tilde{R} + \tilde{K}^2 - \tilde{K}_{ab} \tilde{K}^{ab}) \quad (\text{C3a})$$

$$\tilde{\mathcal{H}}_a^G = -2 \sqrt{\tilde{g}} \tilde{g}^b_a n^c \tilde{G}_{bc} = 2 \sqrt{\tilde{g}} \tilde{D}_b (\tilde{K}^{ab} - \tilde{K} \tilde{g}^{ab}) . \quad (\text{C3b})$$

By introducing the $(s+1)$ -dimensional momentum

$$\tilde{\pi}^{ab} = \frac{\partial \mathcal{L}^G}{\partial (\tilde{\mathcal{L}}_t \tilde{g}_{ab})} = \sqrt{\tilde{g}} (\tilde{K}^{ab} - \tilde{g}^{ab} \tilde{K}) , \quad (\text{C4})$$

the vacuum action $S^G = \int d^{s+2}x \sqrt{-\tilde{g}} \mathcal{L}^G$ can be cast

into the form

$$\begin{aligned} S^G[\hat{g}_{ab}, \hat{\pi}^{ab}; \hat{N}^a, N] &= \int dt \int d^{s+1}x (\hat{\pi}^{ab} \hat{\mathcal{L}}_t \hat{g}_{ab} - N \hat{\mathcal{H}}_{\perp}^G - N^a \hat{\mathcal{H}}_a^G) \\ &\quad - 2 \int d^{s+2}x \sqrt{-\hat{g}} \tilde{\nabla}_a (\alpha^a - \hat{K} n^a) \\ &\quad - 2 \int dt \int d^{s+1}x \hat{D}_a (N_b \hat{\pi}^{ab}) , \end{aligned} \quad (\text{C5})$$

and the constraints transform to

$$\begin{aligned} \hat{\mathcal{H}}_{\perp}^G(\hat{x}; \hat{g}_{ab}, \hat{\pi}^{ab}) &= -\sqrt{\hat{g}} \hat{R} \\ &\quad + \frac{1}{\sqrt{\hat{g}}} \left(\hat{\pi}^{ab} \hat{\pi}_{ab} - \frac{1}{s} \hat{\pi}^2 \right) , \end{aligned} \quad (\text{C6a})$$

$$\hat{\mathcal{H}}_a(\hat{x}; \hat{g}_{ab}, \hat{\pi}^{ab}) = -2 \hat{D}^b \hat{\pi}_{ab} \quad (\text{C6b})$$

for any point $\hat{x} \in S_t$. Extremizing the action (C5) with respect to the canonical variables \hat{g}_{ab} and $\hat{\pi}^{ab}$ yields the field equations

$$\hat{\mathcal{L}}_t \hat{g}_{ab} = 2N \hat{g}^{-1/2} \left(\hat{\pi}_{ab} - \frac{1}{n} \hat{\pi} \hat{g}_{ab} \right) + \hat{\mathcal{L}}_{\mathbf{N}} \hat{g}_{ab} , \quad (\text{C7})$$

$$\begin{aligned} \hat{\mathcal{L}}_t \hat{\pi}^{ab} &= -2N \hat{g}^{-1/2} \left(\hat{\pi}^a{}_c \hat{\pi}^{bc} - \frac{1}{n} \hat{\pi} \hat{\pi}^{ab} \right) \\ &\quad + \frac{1}{2} N \hat{g}^{-1/2} \left(\hat{\pi}_{cd} \hat{\pi}^{cd} - \frac{1}{n} \hat{\pi}^2 \right) \hat{g}^{ab} + \hat{\mathcal{L}}_{\mathbf{N}} \hat{\pi}^{ab} \\ &\quad - N \hat{g}^{1/2} \left(\hat{R}^{ab} - \frac{1}{2} \hat{R} \hat{g}^{ab} \right) \\ &\quad + \hat{g}^{1/2} (\hat{D}^a \hat{D}^b N - \hat{g}^{ab} \hat{D}^c \hat{D}_c N) , \end{aligned} \quad (\text{C8})$$

whereas extremizing the action with respect to the Lagrange multipliers N and \hat{N}^a yields the super-Hamiltonian and the super-momentum constraints of vacuum gravity: $\hat{\mathcal{H}}_{\perp}^G = 0$ and $\hat{\mathcal{H}}_a = 0$.

The dynamical equations (C7) and (C8) can be equally interpreted as the time evolution of the canonical variables, generated by the smeared constraints $\hat{H}^G[N] = \hat{H}_{\perp}^G[N] + \hat{H}_a^G[N^a]$:

$$\hat{\mathcal{L}}_t \hat{g}_{ab} = \{\hat{g}_{ab}, \hat{H}^G[N]\} , \quad (\text{C9a})$$

$$\hat{\mathcal{L}}_t \hat{\pi}^{ab} = \{\hat{\pi}^{ab}, \hat{H}^G[N]\} , \quad (\text{C9b})$$

where the Poisson bracket is defined by

$$\begin{aligned} \{f(\hat{x}), h(\hat{x}')\} &= \int_{S_t} d^{s+1}\hat{x}'' \frac{\delta f(\hat{x})}{\delta \hat{g}_{ab}(\hat{x}'')} \frac{\delta h(\hat{x}')}{\delta \hat{\pi}^{ab}(\hat{x}'')} \\ &\quad - \int_{S_t} d^{s+1}\hat{x}'' \frac{\delta f(\hat{x})}{\delta \hat{\pi}^{ab}(\hat{x}'')} \frac{\delta h(\hat{x}')}{\delta \hat{g}_{ab}(\hat{x}'')} \end{aligned} \quad (\text{C10})$$

for any function $f(\hat{x}; \hat{g}_{ab}, \hat{\pi}^{ab})$ and $g(\hat{x}; \hat{g}_{ab}, \hat{\pi}^{ab})$. The

Poisson brackets of the constraints give the Dirac algebra

$$\begin{aligned} \{\hat{\mathcal{H}}_{\perp}^G(\hat{x}), \hat{\mathcal{H}}_{\perp}^G(\hat{x}')\} &= \hat{g}^{ab}(\hat{x}) \hat{\mathcal{H}}_a^G(\hat{x}) \delta_{,b}(\hat{x}, \hat{x}') \\ &\quad - (\hat{x} \leftrightarrow \hat{x}') , \end{aligned} \quad (\text{C11a})$$

$$\begin{aligned} \{\hat{\mathcal{H}}_{\perp}^G(\hat{x}), \hat{\mathcal{H}}_a^G(\hat{x}')\} &= \hat{\mathcal{H}}^G \perp(\hat{x}) \delta_{,a}(\hat{x}, \hat{x}') \\ &\quad + \hat{\mathcal{H}}_{\perp,a}^G(\hat{x}) \delta(\hat{x}, \hat{x}') , \end{aligned} \quad (\text{C11b})$$

$$\begin{aligned} \{\hat{\mathcal{H}}_a^G(\hat{x}), \hat{\mathcal{H}}_b^G(\hat{x}')\} &= \hat{\mathcal{H}}_b^G(\hat{x}) \delta_{,a}(\hat{x}, \hat{x}') \\ &\quad - (a\hat{x} \leftrightarrow b\hat{x}') . \end{aligned} \quad (\text{C11c})$$

The dynamical equations (C7) and (C8) together with the constraints (C6) provide the full geometrodynamics of the vacuum gravity. Any matter field can be equally $((s+1)+1)$ -dimensionally decomposed and coupled to gravity.

The next step necessary in order to recover the $(s+1+1)$ -dimensional decomposition of geometrodynamics is a further split. We can interpret the momenta π^{ab} , p_a and p introduced in the main text as the projections of $\hat{\pi}^{ab}$:

$$\hat{\pi}^{ab} = \pi^{ab} + M l^{(a} p^{b)} + \frac{M}{2} l^a l^b p . \quad (\text{C12})$$

By inserting Eq. (C12) into the action (C5), we regain the fully decomposed action (18) with the constraints (20). Similarly, the dynamical equations (C7) and (C8) can be split by applying the formula (C12). The only non-trivial step is the decomposition of the Lie-derivatives. For this we employ:

$$\begin{aligned} &(\hat{\mathcal{L}}_t - \hat{\mathcal{L}}_{\mathbf{N}}) \hat{\pi}^{ab} \\ &= \left(\frac{\partial}{\partial t} - \mathcal{L}_{\mathbf{N}} \right) \pi^{ab} + M l^{(a} \left(\frac{\partial}{\partial t} - \mathcal{L}_{\mathbf{N}} \right) p^{b)} \\ &\quad + \frac{M}{2} l^a l^b \left(\frac{\partial}{\partial t} - \mathcal{L}_{\mathbf{N}} \right) p - \frac{NM}{\sqrt{g}} (p^a p^b + p l^{(a} p^{b)}) \\ &\quad - \frac{N}{2s\sqrt{g}} \left(\frac{s-1}{2} M p^2 - \pi p \right) l^a l^b . \end{aligned} \quad (\text{C13})$$

With this we regain the fully decomposed dynamical equations (29a)-(30c). From Eqs. (C12) and (C6) we can also obtain the constraints (20), by using the twice contracted Gauss equation for the hypersurface $\Sigma_{t\chi}$ of \mathcal{S}_t :

$$\hat{R} = R + L^2 - L_{ab} L^{ab} + 2 \hat{D}_a (l^c \hat{D}_c l^a - L l^a) . \quad (\text{C14})$$

We have thus shown that the variation principle and the $(s+1+1)$ -decomposition commute, i.e. no matter which order we apply the decomposition of the bulk and the extremization of the action, the result is the same.

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